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Comparisons between harmonic balance and nonlinear output frequency response function in nonlinear system analysis

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Abstract

By using the Duffing oscillator as a case study, this paper shows that the harmonic components in the nonlinear system response to a sinusoidal input calculated using the nonlinear output frequency response functions (NOFRFs) are one of the solutions obtained using the harmonic balance method (HBM). A comparison of the performances of the two methods shows that the HBM can capture the well-known *jump phenomenon*, but is restricted by computational limits for some strongly nonlinear systems and can fail to provide accurate predictions for some harmonic components. Although the NOFRFs cannot capture the *jump phenomenon*, the method has few computational restrictions. For the nonlinear damping systems, the NOFRFs can give better predictions for all the harmonic components in the system response than the HBM even when the damping system is strongly nonlinear.

1. Introduction

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. Various approaches, including the perturbation method [1–6], multiple scale method [7–12], and the harmonic balance method (HBM) [12–23] have been developed to study the forced periodic motions of these nonlinear oscillators. Among these methods, the HBM is considered to be one of powerful methods capable of handling strongly nonlinear behaviours and, it can converge to an accurate periodic solution for smooth nonlinear systems [13].

The HBM method is based on the assumption that the system time domain response can be expressed in the form of a Fourier series. Therefore, the HBM is usually used to study nonlinear systems where the output responses of which are periodic in time. Such nonlinear systems range from models as simple as the Duffing oscillator [14] to more complex models such as cracked rotors [15]. More applications of the HBM can be found in the study of the nonlinear response of airfoils [16–17], nonlinear conservative systems [18], hysteretic

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two-degree-of-freedom systems [19], the third-order (jerk) differential equations [20] and the Jeffcott rotor [21]. By using the HBM, some interesting phenomena unique to nonlinear systems have been observed, among which the most well known is *jump phenomenon* where the response amplitude of a nonlinear oscillator changes suddenly at some critical value of the frequency of the excitation [13]. Although the basic idea of the HBM is quite simple (to substitute a Fourier series form solution of the system time domain response into the governing equations of the system under study, and to equate coefficients of the same harmonic components), its implementation is actually not easy [14]. First, if many frequency components are taken into account in the HBM to reach accurate results, it is highly possible for the HBM to fail. Second, for the Duffing oscillator, the HBM is typically easy to implement but, for models with more complex nonlinearities, it may be very difficult or impossible to implement. Moreover, it is always necessary to write specific computation programs for different nonlinear models [14], and that is why improved HBM need to be developed.

The Volterra series approach [22–24] is another powerful method for the analysis of nonlinear systems, which extends the well-known concept of the convolution integral for linear systems to a series of multidimensional convolution integrals. The Fourier transforms of the Volterra kernels, called generalised frequency response functions (GFRFs) [25], are an extension of the linear frequency response function (FRF) to the nonlinear case. If a differential equation or discrete-time model is available for a nonlinear system, the GFRFs can be determined using the algorithm in Refs. [26–28]. However, the GFRFs are multidimensional functions [29,30], which can be much more complicated than the linear FRF and can be difficult to measure, display and interpret in practice. Recently, a novel concept known as nonlinear output frequency response functions (NOFRFs) was proposed by the authors [31]. The concept can be considered to be an alternative extension of the classical FRF for linear systems to the nonlinear case. NOFRFs are one-dimensional functions of frequency, which allows the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems and provides great insight into the mechanisms which dominate many nonlinear behaviours. For a nonlinear system subjected to a harmonic input, the response could also be described by a Fourier series using the NOFRFs. The present study is concerned with a comparison study between the NOFRFs and HBM methods in the analysis of a class of nonlinear systems.

2. Harmonic balance method (HBM)

In the HBMs [14], the solution of a nonlinear system is assumed to be of the form of a truncated Fourier series:

$$y(t) = d_0 + \sum_{n=1}^{\bar{N}} (a_n \cos(n\omega t) + b_n \sin(n\omega t)), \tag{1}$$

where d_0, a_n and b_n $(n = 1, \ldots, \bar{N})$ are known as the HB solution Fourier coefficients, and $\bar{N}$ the number of harmonic components used in the HB-truncated Fourier series expansion. The principle of the HBM can be illustrated using the Duffing oscillator:

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 y^3 = A \cos(\omega t), \tag{2}$$

where m, c, k_1 and k_3 are the parameters of the mass, damping and stiffness of the system respectively. A and ω are the external excitation force amplitude and frequency of the oscillator.

The Fourier expansions of the first and second derivatives of the output of system (2) are:

$$\dot{y}(t) = \sum_{n=1}^{\bar{N}} n\omega(-a_n \sin(n\omega t) + b_n \cos(n\omega t)), \tag{3}$$

$$\ddot{y}(t) = \sum_{n=1}^{\bar{N}} -n^2\omega^2(a_n \cos(n\omega t) + b_n \sin(n\omega t)). \tag{4}$$

The Fourier expansion of the cubic term of the output $y(t)$ in Eq. (2) can be expressed, when retaining $\bar{N}$ harmonic components, as

$$(y(t))^3 = \bar{d}_0 + \sum_{n=1}^{\bar{N}} (\bar{a}_n \cos(n\omega t) + \bar{b}_n \sin(n\omega t)), \tag{5}$$

where

$$\bar{d}_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left(d_0 + \sum_{n=1}^{\overline{N}} (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \right)^3 \mathrm{d}t, \tag{6}$$

$$\bar{a}_n = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left(d_0 + \sum_{n=1}^{\bar{N}} (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \right)^3 \cos(n\omega t)\, \mathrm{d}t, \tag{7}$$

$$\bar{b}_n = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left(d_0 + \sum_{n=1}^{\bar{N}} (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \right)^3 \sin(n\omega t)\, \mathrm{d}t. \tag{8}$$

Substituting Eqs. (1) and (3)–(8) into Eq. (2), and equating coefficients associated with each harmonic components $\cos(n\omega t)$ and $\sin(n\omega t)(n = 0, 1, \ldots, \bar{N})$ yields $2\bar{N} + 1$ equations:

$$\left.\begin{array}{l} k_1 d_0 + k_3 \bar{d}_0 = 0 \\ -m\omega^2 a_1 + c\omega b_1 + k_1 a_1 + k_3 \bar{a}_1 = A \\ -m\omega^2 b_1 - c\omega a_1 + k_1 b_1 + k_3 \bar{b}_1 = 0 \\ \qquad \vdots \\ -mN^2\omega^2 a_N + cN\omega b_N + k_1 a_N + k_3 \bar{a}_N = 0 \\ -mN^2\omega^2 b_N - cN\omega a_N + k_1 b_N + k_3 \bar{b}_N = 0 \end{array}\right\}. \tag{9}$$

Solving Eq. (9) requires the analytical expressions for the nonlinear functions $\bar{d}_0$, $\bar{a}_n$, $\bar{b}_n(n = 1, \ldots, \bar{N})$ in terms of d_0, a_n and b_n $(n = 1, \ldots, \bar{N})$. When using only the fundamental harmonic component, i.e., $\bar{N} = 1$, the HBM is often referred to as the HB1 method. In the case of HB1, it can be deduced that Eq. (9) can be written as

$$-m\omega^2 a_1 + c\omega b_1 + k_1 a_1 + \tfrac{1}{2} k_3 [3a_1^3/2 + 3a_1 b_1^2/2] = A, \tag{10}$$

$$-m\omega^2 b_1 - c\omega a_1 + k_1 b_1 + \tfrac{1}{2} k_3 [3b_1 a_1^2/2 + 3b_1^3/2] = 0, \tag{11}$$

The forms of Eqs. (10) and (11) are relatively simple, and solving them will take only a few seconds using contemporary powerful numerical software routines. However, if the nonlinearity of the Duffing oscillator is strong, the high-order harmonic components may have a considerable effect and can contribute significantly to the whole solution, consequently, a truncated Fourier series expansion may make the solution less accurate. On the other hand, if more harmonic components are considered for the analysis, then Eq. (9) can become quite complex. For example, when the third-order harmonic component is taken into account, i.e., $\bar{N} = 3$, then Eq. (9) is given by

$$-m\omega^2 a_1 + c\omega b_1 + k_1 a_1 + \frac{1}{2} k_3 \begin{bmatrix} 3a_1^3/2 + 3a_1 b_1^2/2 + 3a_1 a_3^2 + 3a_1 b_3^2 \\ 3a_1^2 a_3/2 + 3a_1 b_1 b_3 - 3b_1^2 a_3/2 \end{bmatrix} = A, \tag{12}$$

$$-m\omega^2 b_1 - c\omega a_1 + k_1 b_1 + \frac{1}{2} k_3 \begin{bmatrix} 3b_1 a_1^2/2 + 3b_1^3/2 + 3b_1 a_3^2 + 3b_1 b_3^2 \\ -3b_1 a_1 a_3 - 3b_1^2 b_3/2 + 3a_1^2 b_3/2 \end{bmatrix} = 0, \tag{13}$$

$$-9m\omega^2 a_3 + 3c\omega b_3 + k_1 a_3 + \frac{1}{2}k_3 \begin{bmatrix} a_1^3/2 - 3a_1 b_1^2/2 + 3a_1^2 a_3 + 3b_1^2 a_3 \\ 3a_3^3/2 + 3a_3 b_3^2/2 \end{bmatrix} = 0, \tag{14}$$

$$-9m\omega^2 b_3 - 3c\omega a_3 + k_1 b_3 + \frac{1}{2}k_3 \begin{bmatrix} -b_1^3/2 + 3b_1 a_1^2/2 + 3b_1^2 b_3 + 3a_1^2 b_3 \\ 3a_3^2 b_3/2 + 3b_3^3/2 \end{bmatrix} = 0, \tag{15}$$

which is obviously much more complicated than the case of $\bar{N} = 1$. In the case of $\bar{N} = 3$, the HBM is referred to as the HB3. In most cases, the software cannot find solutions for Eqs. (12)–(15) because of the complex forms. To obtain a solution, therefore, many terms have to be ignored, for example, $3a_3^3/2$ and $3a_3b_3^2/2$ in (14) and $3a_3^2b_3/2$ and $3b_3^3/2$ in (15). This is a common practice when using HBMs to conduct nonlinear system analysis. Usually, more than one solution, some of which might involve complex values, can be found for the HBMs. Only the real valued solutions are physically meaningful for the underlying problem.

3. Nonlinear output frequency response functions (NOFRFs)

The definition of NOFRFs is based on the Volterra series theory of nonlinear systems. Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series [32,33]:

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i)\, \mathrm{d}\tau_i, \tag{16}$$

where $y(t)$ and $u(t)$ are the output and input of the system, $h_n(\tau_1, \ldots, \tau_n)$ is the nth-order Volterra kernel, and N denotes the maximum order of the system nonlinearity. Lang and Billings [25] derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} Y(\mathrm{j}\omega) = \sum\limits_{n=1}^{N} Y_n(\mathrm{j}\omega) \quad \text{for} \quad \forall\omega, \\ Y_n(\mathrm{j}\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n) \prod\limits_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}. \end{cases} \tag{17}$$

In (2), $Y(\mathrm{j}\omega)$ is the spectrum of the system output, $Y_n(\mathrm{j}\omega)$ represents the nth-order output frequency response of the system:

$$H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \mathrm{e}^{-(\omega_1\tau_1+,\ldots,+\omega_n\tau_n)j} \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \tag{18}$$

is the nth-order GFRF [25], and

$$\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n) \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}$$

denotes the integration of $H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n)\prod_{i=1}^{n} U(\mathrm{j}\omega_i)$ over the n-dimensional hyper-plane $\omega_1 + \cdots + \omega_n = \omega$.

The new concept of the NOFRFs recently proposed by Lang and Billings [31] is defined as

$$G_n(\mathrm{j}\omega) = \frac{\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n)\prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}}{\int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega}} \tag{19}$$

under the condition that

$$U_n(\mathrm{j}\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\, \mathrm{d}\sigma_{n\omega} \neq 0. \tag{20}$$

Notice that $G_n(\mathrm{j}\omega)$ is valid over the frequency range of $U_n(\mathrm{j}\omega)$, which can be determined using the algorithm in Refs. [25,34].

By introducing the NOFRFs $G_n(j\omega)$, $n = 1, \ldots N$, Eq. (17) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega)U_n(j\omega), \tag{21}$$

which is similar to the description of the output frequency response for linear systems. The NOFRFs reflect a combined contribution of the system and the input to the system output frequency response behaviour.

When the input is a sinusoidal force:

$$u(t) = A\cos(\omega_F t + \beta). \tag{22}$$

Peng et al. [35] have showed that the frequency components of the nth-order output $Y_n(j\omega)$ of the nonlinear system can be determined as

$$\Omega_n = \{(-n + 2k)\omega_F, \quad k = 0, 1, \ldots, n\} \tag{23}$$

and the frequency components of the system output $Y(j\omega)$ can be determined as

$$\Omega = \bigcup_{n=1}^{N} \Omega_n = \{k\omega_F, \quad k = -N, \ldots, -1, 0, 1, \ldots, N\} \tag{24}$$

and the kth superharmonic component of the system output can be expressed as

$$Y(jk\omega_F) = \sum_{n=1}^{[(N-k+1)/2]} G^H_{k+2(n-1)}(jk\omega_F)A_{k+2(n-1)}(jk\omega_F) \quad (k = 0, 1, \ldots, N), \tag{25}$$

where [.] means to take the integer part, and

$$A_n(j(-n + 2k)\omega_F) = \frac{1}{2^n}\frac{n!}{k!(n-k)!}|A|^n e^{j(-n+2k)\beta}, \tag{26}$$

$$G^H_n(j(-n + 2k)\omega_F) = H_n(\overbrace{j\omega_F, \ldots, j\omega_F}^{k}, \overbrace{-j\omega_F, \ldots, -j\omega_F}^{n-k}). \tag{27}$$

Notice $H_n(j\omega_1, \ldots, j\omega_n)$ is a symmetric function, thus, for the case of sinusoidal inputs, $G^H_n(j\omega)$ over the nth-order output frequency range $\Omega_n = \{(-n + 2k)\omega_F, \quad k = 0, 1, \ldots, n\}$ is equal to the $H_n(j\omega_1, \ldots, j\omega_n)$ evaluated at $\omega_1 = \cdots = \omega_k = \omega_F, \omega_{k+1} = \cdots = \omega_n = -\omega_F$, $(k = 0, \ldots, n)$.

Using the algorithm by Billings and Peyton Jones [27,28] and Eq. (27), the NOFRFs of the Duffing oscillator under a harmonic input can be obtained. The results show that all even order NOFRFs are zero; for the first and third harmonic components in the output, the NOFRFs up to the seventh-order are as follows:

$$G^H_1(j\omega_F) = \frac{1}{-m\omega_F^2 + jc\omega_F + k_1}, \tag{28}$$

$$G^H_3(j\omega_F) = -k_3 G^H_1(-j\omega_F)G^H_1(j\omega_F)G^H_1(j\omega_F)G^H_1(j\omega_F), \tag{29}$$

$$G^H_3(j3\omega_F) = -k_3 G^H_1(j\omega_F)G^H_1(j\omega_F)G^H_1(j\omega_F)G^H_1(j3\omega_F), \tag{30}$$

$$G^H_5(j\omega_F) = -\frac{3}{10}k_3 G^H_1(j\omega_F)\begin{bmatrix} 3G^H_3(-j\omega_F)G^H_1(j\omega_F)G^H_1(j\omega_F) \\ +6G^H_3(j\omega_F)G^H_1(-j\omega_F)G^H_1(j\omega_F) \\ +G^H_3(j3\omega_F)G^H_1(-j\omega_F)G^H_1(-j\omega_F) \end{bmatrix}, \tag{31}$$

$$G^H_5(j3\omega_F) = -\frac{3}{10}k_3 G^H_1(j3\omega_F)\begin{bmatrix} 6G^H_3(j\omega_F)G^H_1(j\omega_F)G^H_1(j\omega_F) \\ +4G^H_3(j3\omega_F)G^H_1(-j\omega_F)G^H_1(j\omega_F) \end{bmatrix}, \tag{32}$$

$$G_7^H(\mathrm{j}\omega_F) = -k_3 G_1^H(\mathrm{j}\omega_F)\left\{\begin{array}{l} \dfrac{1}{7}\left[\begin{array}{l} 12G_5^H(\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F) \\ \quad +3G_5^H(\mathrm{j}3\omega_F)G_1^H(-\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F) \\ \quad +6G_5^H(-\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F) \end{array}\right] \\ +\dfrac{3}{70}\left[\begin{array}{l} 24G_3^H(\mathrm{j}3\omega_F)G_3^H(-\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F) \\ \quad +8G_3^H(\mathrm{j}3\omega_F)G_3^H(-\mathrm{j}3\omega_F)G_1^H(\mathrm{j}\omega_F) \\ \quad +36G_3^H(\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F) \\ \quad +72G_3^H(\mathrm{j}\omega_F)G_3^H(-\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F) \end{array}\right] \end{array}\right\}, \tag{33}$$

$$G_7^H(\mathrm{j}3\omega_F) = -k_3 G_1^H(\mathrm{j}3\omega_F)\left\{\begin{array}{l} \dfrac{1}{7}\left[\begin{array}{l} 10G_5^H(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F) \\ \quad +G_5^H(\mathrm{j}5\omega_F)G_1^H(-\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F) \\ \quad +10G_5^H(\mathrm{j}3\omega_F)G_1^H(-\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F) \end{array}\right] \\ +\dfrac{3}{70}\left[\begin{array}{l} 60G_3^H(\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F) \\ \quad +40G_3^H(\mathrm{j}3\omega_F)G_3^H(-\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F) \\ \quad +40G_3^H(\mathrm{j}3\omega_F)G_3^H(\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F) \end{array}\right] \end{array}\right\}. \tag{34}$$

Based on equations (25), (28)–(34), the harmonic components up to third order in the output of the Duffing oscillator can be determined when the oscillator's output response to a harmonic input can be approximated by a Volterra series expansion up to seventh order.

Eq. (25) provides a straightforward way to express the response of nonlinear systems subjected to a harmonic input using the NOFRFs. The focus of the present study is dedicated to a comparison study between the NOFRFs and the HBM in the analysis of a class of nonlinear systems. First, the Duffing oscillator is used as a case study to reveal the relationships between the NOFRFs and the HBM, and then numerical examples will be used to compare the performances of the methods in nonlinear system analysis.

4. Relationships between the HBM and NOFRFs

Theoretically, the output spectrum of a nonlinear system such as the Duffing oscillator has to be expressed using an infinite Volterra series. Therefore, ideally, equation (25) should be expressed as

$$Y(\mathrm{j}k\omega_F) = \sum_{n=1}^{\infty} G_{k+2(n-1)}^H(\mathrm{j}k\omega_F)A_{k+2(n-1)}(\mathrm{j}k\omega_F) \quad (k = 0, 1, \ldots, \infty). \tag{35}$$

However, in practice a truncated series can be used provided the number of terms included can give an accurate approximation to the response of the system. For the Duffing oscillator, without loss of generality, it is assumed in the following analysis that, in Eq. (21), the first three terms are sufficient to approximate the system response and the effect of the higher-order terms on the response is negligible. Under this assumption, it is known from (26) that

$$\begin{aligned} &Y(\mathrm{j}0) = G_2^H(\mathrm{j}0)A_2(\mathrm{j}0) + G_4^H(\mathrm{j}0)A_4(\mathrm{j}0) + \cdots = 0, \\ &Y(\mathrm{j}\omega_F) = \sum_{n=1}^{\infty} G_{2n-1}^H(\mathrm{j}\omega_F)A_n(\mathrm{j}\omega_F) \approx G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F) + G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F), \\ &Y(\mathrm{j}2\omega_F) = G_2^H(\mathrm{j}2\omega_F)A_2(\mathrm{j}2\omega_F) + G_4^H(\mathrm{j}2\omega_F)A_4(\mathrm{j}2\omega_F) + \cdots = 0, \\ &Y(\mathrm{j}3\omega_F) = \sum_{n=2}^{\infty} G_{2n-1}^H(\mathrm{j}3\omega_F)A_n(\mathrm{j}3\omega_F) \approx G_3^H(\mathrm{j}3\omega_F)A_3(\mathrm{j}3\omega_F), \\ &Y(-\mathrm{j}\omega_F) \approx G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F) + G_3^H(-\mathrm{j}\omega_F)A_3(-\mathrm{j}\omega_F), \\ &Y(-\mathrm{j}2\omega_F) = 0, \\ &Y(-\mathrm{j}3\omega_F) \approx G_3^H(-\mathrm{j}3\omega_F)A_3(-\mathrm{j}3\omega_F). \end{aligned} \tag{36}$$

Proposition. For nonlinear systems subjected to a sinusoidal input, the harmonic components in the output response determined using the NOFRFs from Eq. (35) are one of the solutions obtained using the HBM.

For simplicity of explanation, the proof of this proposition is demonstrated using the Duffing oscillator as follows. From (36), the response of the Duffing oscillator to a harmonic input can be written as

$$y(t) = Y(\mathrm{j}\omega_F)\exp(\mathrm{j}\omega_F t) + Y(\mathrm{j}3\omega_F)\exp(\mathrm{j}3\omega_F t) + Y(-\mathrm{j}\omega_F)\exp(-\mathrm{j}\omega_F t) + Y(-\mathrm{j}3\omega_F t)\exp(-\mathrm{j}3\omega_F t) + e(t), \quad (37)$$

where $e(t)$ is the truncation error.

Substituting (37) into (2) and extracting the coefficients of the harmonic component of frequency ω_F on the left side of Eq. (2) yields

$$\begin{aligned}
\Gamma_1(\mathrm{j}\omega_F) = {} & \frac{1}{G_1^H(\mathrm{j}\omega_F)}(G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F) + G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F)) \\
& + k_3(3G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F) \\
& + 3G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F)G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F)G_3^H(-\mathrm{j}\omega_F)A_3(-\mathrm{j}\omega_F) \\
& + 6G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F) \\
& + 3G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F)G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F)G_3^H(\mathrm{j}3\omega_F)A_3(\mathrm{j}3\omega_F) \\
& + 3G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F) \\
& + 6G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F)G_3^H(-\mathrm{j}\omega_F)A_3(-\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F) \\
& + 6G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F)G_3^H(-\mathrm{j}\omega_F)A_3(-\mathrm{j}\omega_F)G_3^H(\mathrm{j}3\omega_F)A_3(\mathrm{j}3\omega_F) \\
& + 6G_1^H(-\mathrm{j}\omega_F)A_1(-\mathrm{j}\omega_F)G_3^H(-\mathrm{j}3\omega_F)A_3(-\mathrm{j}3\omega_F)G_3^H(\mathrm{j}3\omega_F)A_3(\mathrm{j}3\omega_F) \\
& + 3G_3^H(-\mathrm{j}\omega_F)A_3(-\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F)G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F) \\
& + 3G_3^H(-\mathrm{j}\omega_F)A_3(-\mathrm{j}\omega_F)G_3^H(-\mathrm{j}\omega_F)A_3(-\mathrm{j}\omega_F)G_3^H(\mathrm{j}3\omega_F)A_3(\mathrm{j}3\omega_F)).
\end{aligned} \quad (38)$$

Obviously, if the harmonic components determined using the NOFRFs are one of the solutions obtained using the HBM, $\Gamma_1(\mathrm{j}\omega_F)$ should be equal to $A_1(\mathrm{j}\omega_F)$.

According to Eqs. (26) and (29), it can be deduced that the third term in (38) is equal to $-G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F)$. Similarly, from Eqs. (31) and (26), it can also be known that the sum of the 4th, 5th and 6th terms in (38) is equal to $-G_5^H(\mathrm{j}\omega_F)A_5(\mathrm{j}\omega_F)$, and from Eqs. (33) and (26), it can be deduced that the 7th, 8th, 9th and 10th terms are a part of $-G_7^H(\mathrm{j}\omega_F)A_7(\mathrm{j}\omega_F)$. In fact, the 11th and 12th terms can be also deduced to be a part of $-G_9^H(\mathrm{j}\omega_F)A_9(\mathrm{j}\omega_F)$. Therefore, compared to the terms $G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F)$ and $G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F)$, the contributions of $G_5^H(\mathrm{j}\omega_F)A_5(\mathrm{j}\omega_F)$, $G_7^H(\mathrm{j}\omega_F)A_7(\mathrm{j}\omega_F)$, and $G_9^H(\mathrm{j}\omega_F)A_9(\mathrm{j}\omega_F)$ to the whole response are negligible, therefore Eq. (38) can be simplified to

$$\begin{aligned}
G_1^H(\mathrm{j}\omega_F)\Gamma_1(\mathrm{j}\omega_F) &= G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F) + G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F) - G_3^H(\mathrm{j}\omega_F)A_3(\mathrm{j}\omega_F) + \Delta_5(\mathrm{j}\omega_F) \\
&= G_1^H(\mathrm{j}\omega_F)A_1(\mathrm{j}\omega_F) + \Delta_5(\mathrm{j}\omega_F),
\end{aligned} \quad (39)$$

where $\Delta_5(\mathrm{j}\omega_F)$ is a negligible error caused by ignoring the insignificant terms. Therefore, $\Gamma_1(j\omega_F) = A_1(\mathrm{j}\omega_F)$, if $\Delta_5(\mathrm{j}\omega_F) = 0$.

Following the same procedure, it can be proved that, for other harmonic components of frequencies $-\omega_F$, $3\omega_F$ and $-3\omega_F$, the coefficients are also balanced. Therefore, it is demonstrated that the proposition holds for the Duffing oscillator.

The proposition reveals the relationship between the NOFRF method and the HBM. In fact, from the perspective of practical applications, the proposition implies that, if a nonlinear system response to a harmonic input can be expanded as a convergent Volterra series, the harmonic components calculated by the NOFRFs using Eq. (35) can approximate one solution of the HBM to an arbitrary accuracy if a sufficient number of terms are taken into account.

5. Numerical studies and discussions

5.1. Case studies

In this section, the performances of the NOFRF and the HBM in nonlinear system analysis will be compared via two case studies where two different nonlinear oscillators including a nonlinear stiffness oscillator and a nonlinear damping oscillator are used to conduct the comparison studies.

5.1.1. Case 1: nonlinear stiffness oscillator studies

For the nonlinear stiffness oscillator, the well-known Duffing oscillator given by Eq. (1) is adopted, which can be rewritten as

$$\ddot{y} + 2\mu\omega_0\dot{y} + \omega_0^2 y + \varepsilon_3\omega_0^6 y^3 = A_0 \cos(\omega_F t), \tag{40}$$

where $\mu = c/\left(2\sqrt{km}\right)$, $\omega_0 = \sqrt{k/m}$, $\varepsilon_3 = k_3/k_1^3$, $A_0 = A/m$. Denote

$$\tau = \omega_0 t; \quad \gamma = \omega_F/\omega_0; \quad \eta = \omega_0^2 y/A_0; \quad \rho = A_0^2\varepsilon_3,$$

then Eq. (40) can be written in the following non-dimensional form:

$$\ddot{\eta} + 2\mu\dot{\eta} + \eta + \rho\eta^3 = \cos(\gamma\tau). \tag{41}$$

For this case study, the values of the used parameters in Eq. (40) are $\mu = 0.04$, $\omega_0 = 12\pi$, $\varepsilon_3 = 0.1$, and the corresponding non-dimensional equation is

$$\ddot{\eta} + 0.08\dot{\eta} + \eta + 0.1A_0^2\eta^3 = \cos(\gamma\tau). \tag{42}$$

It can be seen that the nonlinearity strength of the Duffing oscillator is determined by the value of the coefficient ρ, which is dependent on both the coefficient ε_3 and the excitation level A_0. Duffing oscillators with such strong nonlinearities have been studied using Volterra series in [36,37]. Practical systems with such strong nonlinearities can also be found in [38] where a torsional spring with a strong nonlinear cubic stiffness was used to join two Euler–Bernoulli beams together.

First, consider the case where the amplitude of the sinusoidal input is 1 ($A_0 = 1$) and the range of ω_F is $\omega_0/10 \leqslant \omega_F \leqslant 3\omega_0$. The HB3 method was used to calculate the first harmonics and the third harmonics by solving Eqs. (12)–(15). In addition, the forced responses of the oscillator under different input frequencies have been calculated using the fourth-order *Runge–Kutta* method, from which the first and third harmonics have also been extracted.

Fig. 1 shows the amplitudes of the first harmonics and the third harmonics obtained by the HB3 method and the *Runge–Kutta* method respectively. For the HB3 method, the amplitudes of the first harmonic and the third harmonic are, respectively, defined as $\sqrt{a_1^2 + b_1^2}$ and $\sqrt{a_3^2 + b_3^2}$. The simulations using the *Runge–Kutta* method clearly show that, around the frequency $\omega_F = 1.5\omega_0$, the well-known *jump phenomenon* has occurred. At the jump point, the amplitudes of the first and third harmonic have suddenly changed. Obviously, the HB3 method has successfully captured the *jump phenomenon*. Over the whole frequency range considered, there always exists one solution obtained using the HB3 method that matches the result by the *Runge–Kutta* method very well for both the first harmonics and the third harmonics.

Fig. 2 shows the comparison between the amplitudes of the first harmonics and the third harmonics obtained by the NOFRF expansion up to the seventh order and the *Runge–Kutta* method, respectively. It can be seen that, apart from the frequency range between $0.7\omega_0$ and $1.5\omega_0$, the results obtained by the NOFRF method match the results by the *Runge–Kutta* method very well. At the frequency range between $0.7\omega_0$ and $1.5\omega_0$, there is a big deviation between the results of the two methods. Actually, at this frequency range, the representation of the Volterra series may be divergent, and the NOFRF expansion thus fails to represent the harmonic components of the responses of the Duffing oscillator in this frequency range.

The problem regarding the convergence of the Volterra series in representing nonlinear systems is very complicated. As far as we are aware, there is no criterion available that is valid for any nonlinear system to judge whether the Volterra series representation is convergent or not. For the Duffing oscillator subjected to

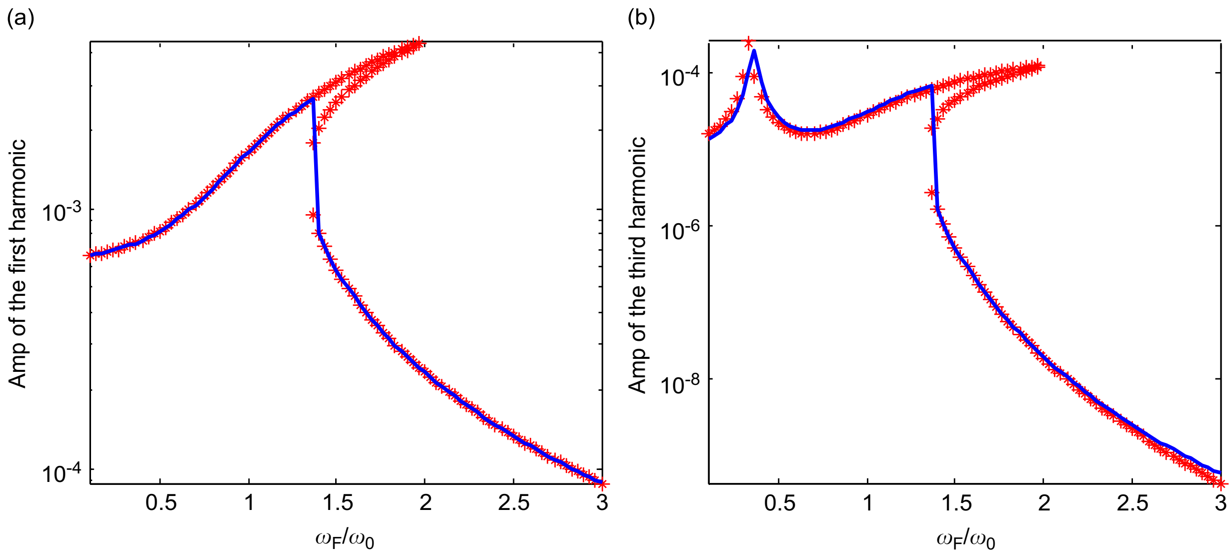

Fig. 1. Comparison between the HB3 and the *Runge–Kutta* method (star-HB3): (a) the first harmonic and (b) the third harmonic.

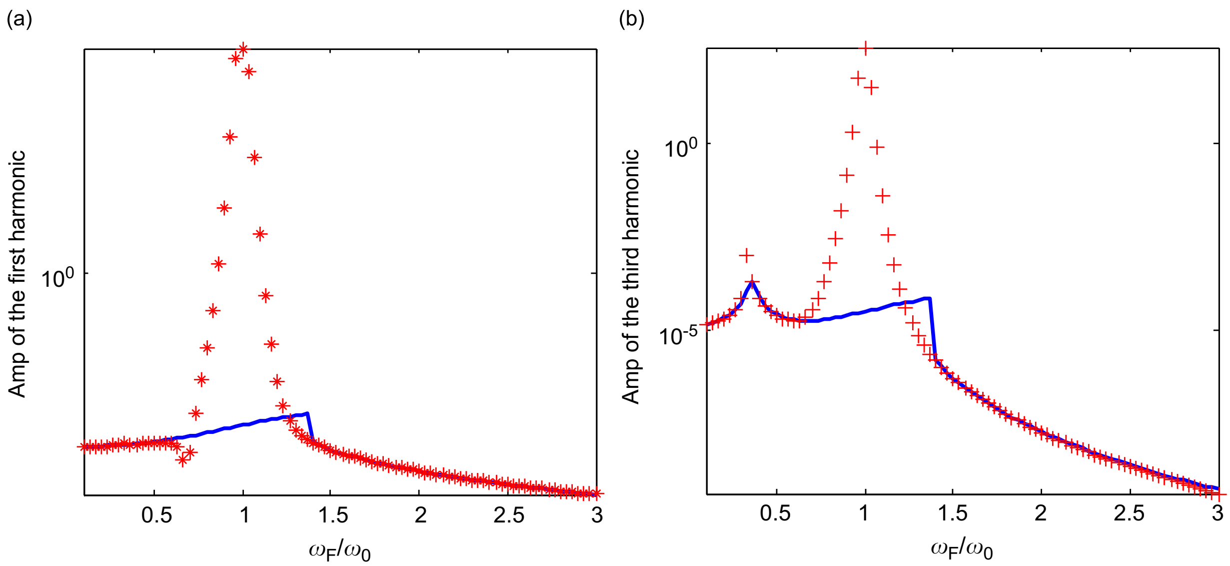

Fig. 2. Comparison between the NOFRF and the *Runge–Kutta* method (star-NOFRFs): (a) the first harmonic and (b) the third harmonic.

harmonic excitation loading, Tomlinson et al. [39] have proposed a simple criterion given as

$$A_0 < \left(\frac{3}{2}\varepsilon\omega_0^2|G_1^H(\mathrm{j}\omega_F)|^3\right)^{-2}. \tag{43}$$

For example, according to the criterion (43), for the Duffing oscillator in Eq. (40), when the frequency of excitation is $0.8\omega_0$, if $A_0 > 0.5709$, then the Volterra series representation is divergent. As the amplitude of the excitation used in the case study is 1.0, which is larger than 0.5709, the Volterra series representation therefore becomes divergent. This makes the NOFRF expansion fail to represent the harmonic components at $\omega_F = 0.8\omega_0$. The failure of NOFRF expansion to represent the harmonics at other frequencies can also be explained in a similar way. In addition, it is worth noting that for the Duffing oscillator the occurrence of the *jump phenomenon* generally implies a divergent Volterra series representation and, therefore, the NOFRFs are not able to capture the *jump phenomenon* in the Duffing oscillator.

Figs. 3 and 4 show comparisons under the situation where the frequency of the input is taken as $\omega_0/2$ but the amplitude A_0 is changed between 0.05 and 3. Fig. 3 gives the comparison between the HB3 method and the *Runge–Kutta* method. It can be seen that, at the region of $A_0<1.2$, the results from the HB3 method match the results of the *Runge–Kutta* method very well, which implies that the HB3 method can predict the motion of the oscillator accurately for the region $A_0<1.2$. However, when $A_0>1.2$, the deviation between the results obtained by the two different methods increases sharply as the amplitude A_0 increase, especially for the third harmonic. It is

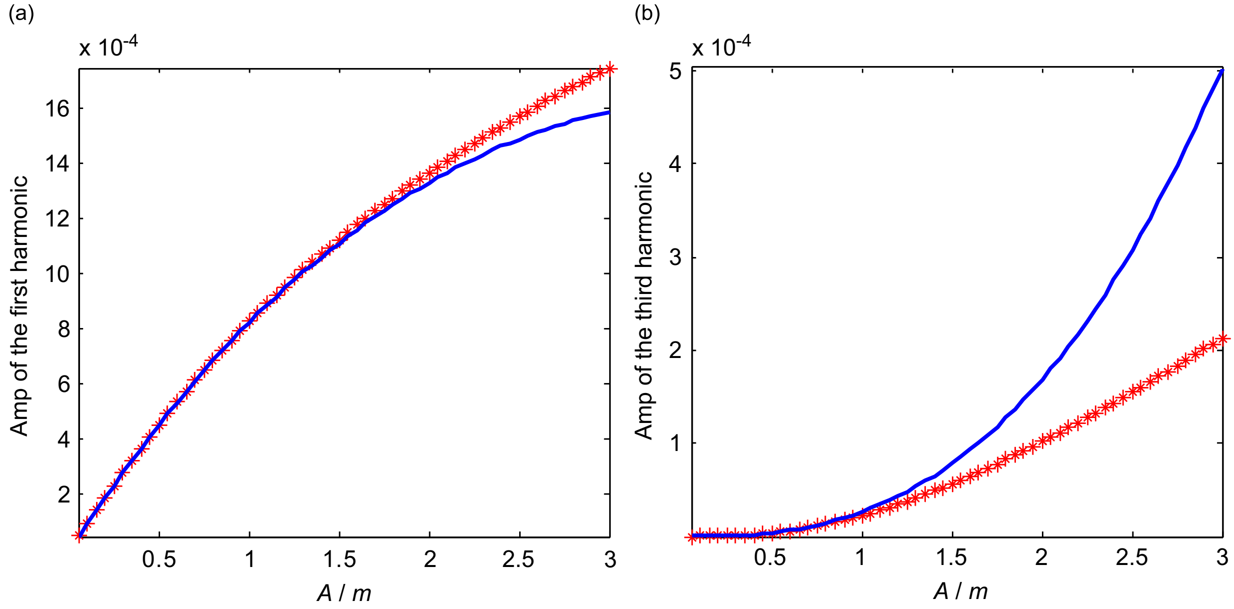

Fig. 3. Comparison between the HB3 and the *Runge–Kutta* method (star-HB3): (a) the first harmonic and (b) the third harmonic.

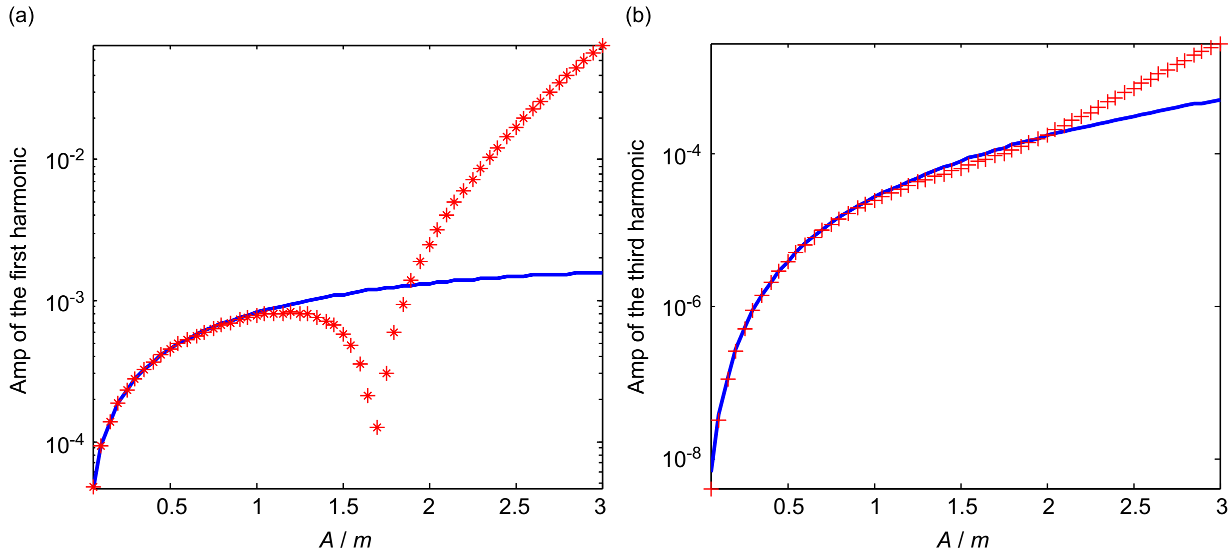

Fig. 4. Comparison between the NOFRFs and the *Runge–Kutta* method (star-NOFRFs): (a) the first harmonic and (b) the third harmonic.

believed that the deviations are introduced by ignoring the other higher-order harmonics, which can contribute significantly to the response of the oscillator when the excitation becomes stronger. Unfortunately, when more harmonics are taken into account in the application of the HBM, the software often fails to find a solution.

Fig. 4 shows the comparison between the NOFRF method and the *Runge–Kutta* method. In the small amplitude region, the NOFRF method can accurately predict the motion of the oscillator. When the frequency of excitation is $\omega_0/2$, according to the criterion (43), if the amplitude of the excitation is larger than 1.6806, the Volterra series representation will be divergent. The results shown in Fig. 4 are basically consistent with the prediction from the criterion.

5.1.2. Case 2: nonlinear damping oscillator studies

The nonlinear damping oscillator is a model that has been widely used to represent shock absorbers [40,41], and is given by

$$m\ddot{y} + c\dot{y} + c_2\dot{y}^2 + c_3\dot{y}^3 + ky = A\cos(\omega t). \quad (44)$$

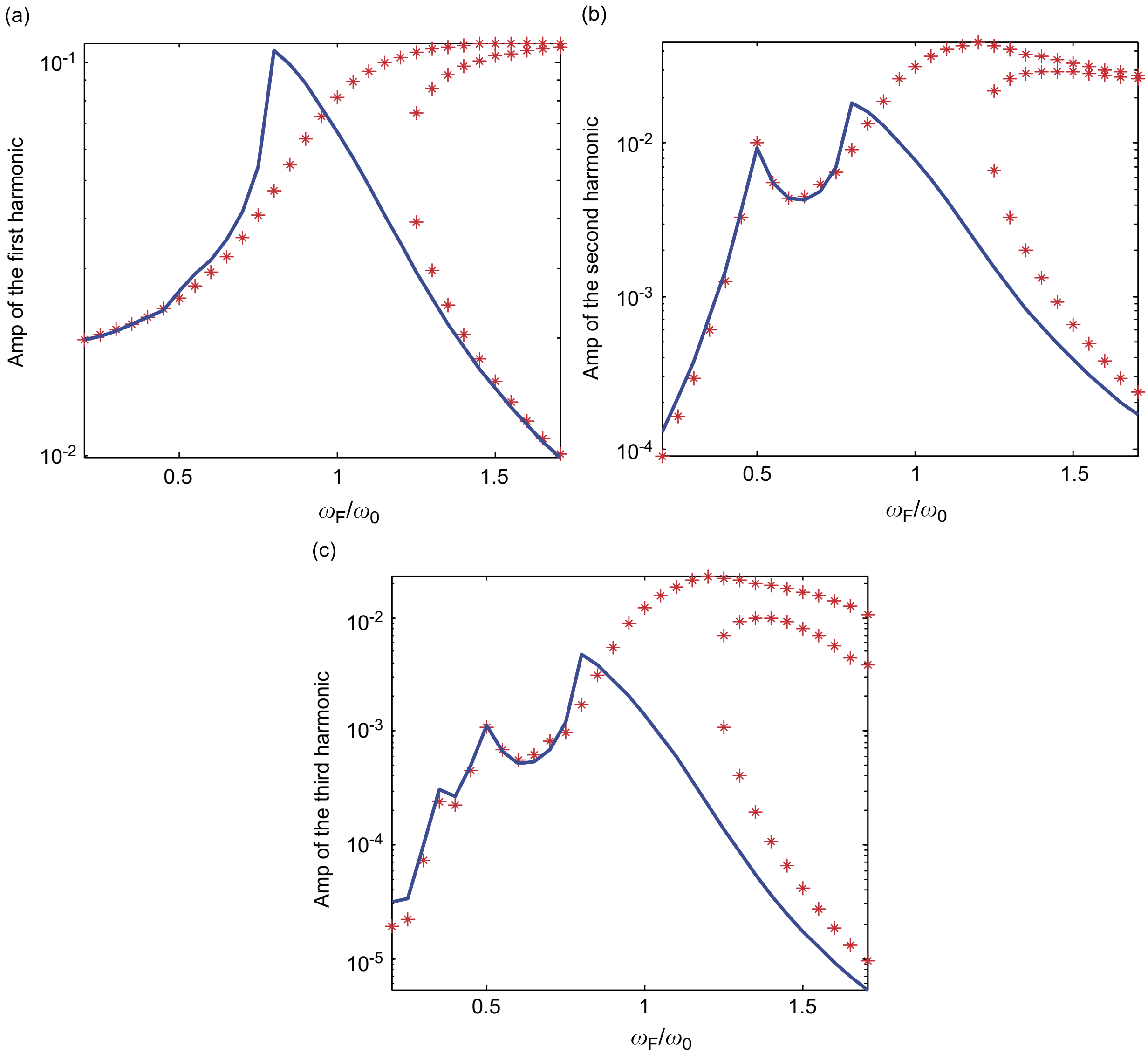

Fig. 5. Comparison between the HB3 and the *Runge–Kutta* method (star-HB3): (a) the first-order harmonic; (b) the second-order harmonic; and (c) the third-order harmonic.

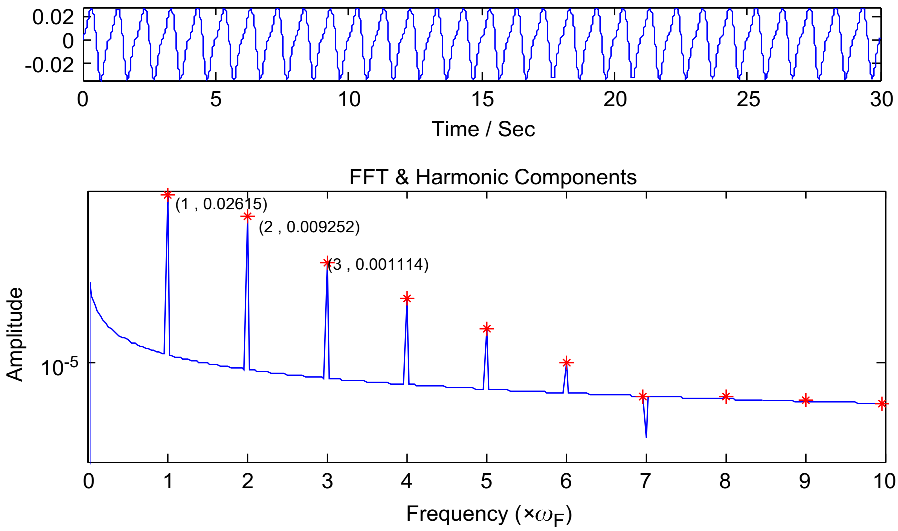

Fig. 6. The time domain response and the spectrum obtained at $\omega_F = 0.5\omega_0$ by the *Runge–Kutta* method.

The values of the system parameters used in this study are $m = 240$, $c = 296$, $c2 = 3000$, $c3 = 800$, $k = 240 \times (4\pi)^2$, $\omega_0 = \sqrt{k/m} = 4\pi$.

Fig. 5 shows the amplitudes of the first, the second and the third harmonics obtained by the HB3 method and the *Runge–Kutta* method, respectively. The amplitude of the input is $A/m = 3$, and the range of ω_F is $0.2\omega_0 \leqslant \omega_F \leqslant 1.7\omega_0$. It can be seen that the results by the HB3 method cannot match the results by the *Runge–Kutta* method well, especially for the second harmonic and the third harmonic. The differences between the results are mainly introduced by ignoring the higher-order harmonics. As indicated by the time domain response and the Fourier spectrum in Fig. 6, which was obtained at $\omega_F = 0.5\omega_0$, the oscillator's behaviour is strongly nonlinear: the other higher-order harmonics are very significant in the spectrum. Ignoring these higher-order harmonics has led to the inaccuracy of the HB3 method.

Fig. 7 shows a comparison between the NOFRF method and the *Runge–Kutta* method for the harmonics up to the third order. Clearly, from Fig. 7 the results by the NOFRF expansions match the results by the *Runge–Kutta* method very well for all harmonics except for a few points around the maximal peaks. This means that the NOFRF expansion can accurately predict the motion of the system (44) in this case. The results shown in Figs. 5 and 7 indicate that the NOFRF up to seventh order is much better than the HBM in the description of the nonlinear damping system.

Figs. 8 and 9 show comparisons under the situations where the frequency of the input is taken as $\omega_0/2$ but the amplitude A/m is changed between 0.2 and 4. It can be seen that, in the region $A/m < 2.5$, the results by the HB3 method match the results of the *Runge–Kutta* method very well, which implies that the HB3 method can predict the motion of the oscillator (44) accurately in the region $A/m < 2.5$. However, when $A/m > 2.5$, the deviation between the results obtained by the two different methods increases with the amplitude A/m for all the first, second and third harmonics. This is because in the region $A/m > 2.5$, the contributions of the other higher-order harmonics to the response of the nonlinear damping oscillator (44) are too significant to be ignored. On the contrary, Fig. 9 shows that, at all amplitudes, the results by the NOFRFs always accurately match the results from the *Runge–Kutta* method, especially for the first harmonic. The results shown in Figs. 8 and 9 have again indicated that the NOFRF methods give a better performance compared to the HBM in the analysis of the nonlinear damping oscillator.

5.2. Discussion

From the above analyses for two kinds of nonlinear oscillators, it can be seen that the HBM and NOFRF each have advantages and drawbacks in nonlinear system analysis. The HBM can capture the *jump*

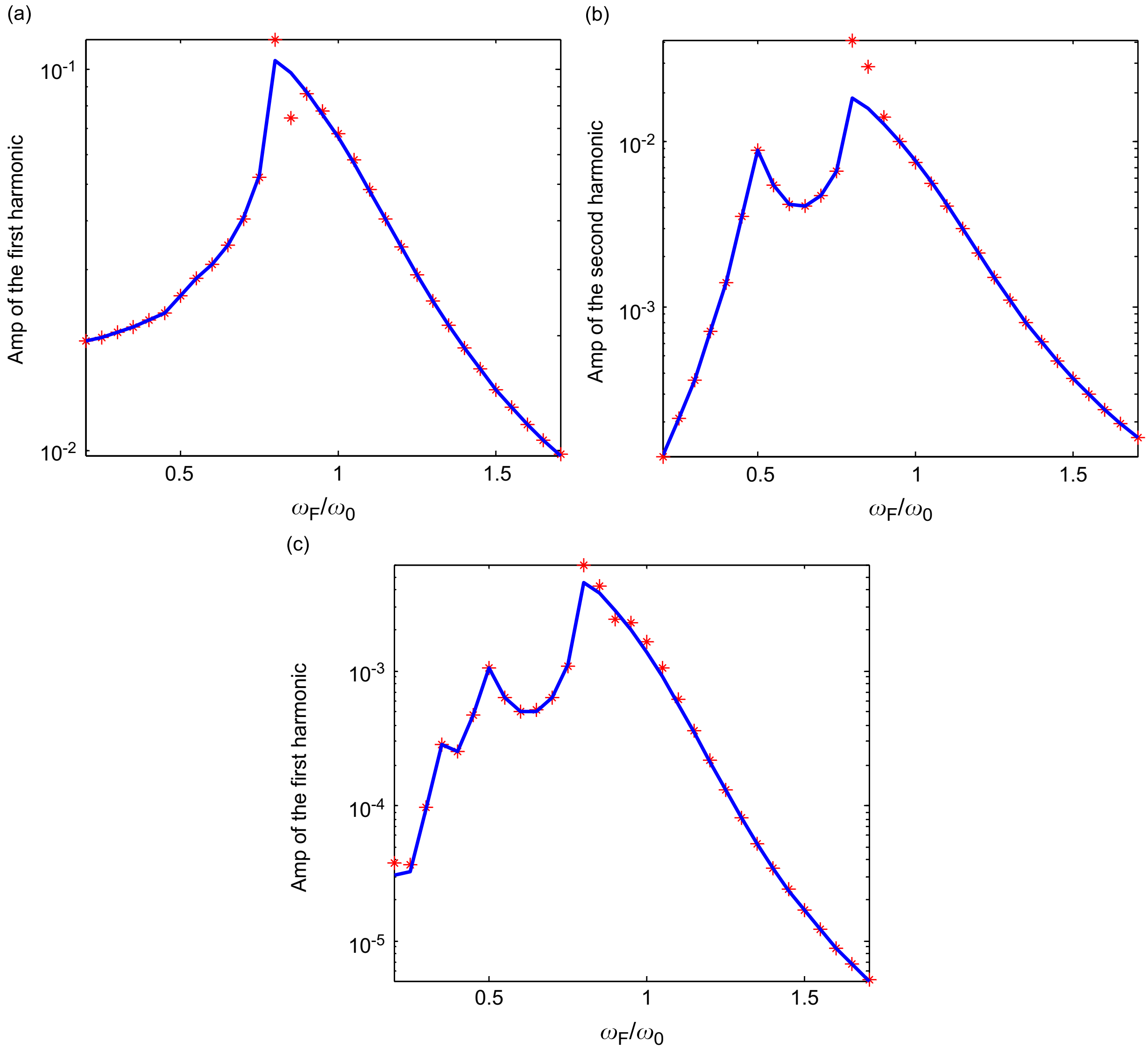

Fig. 7. Comparison between the NOFRF and the *Runge–Kutta* method (star-NOFRFs): (a) the first-order harmonic; (b) the second-order harmonic; and (c) the third-order harmonic.

phenomenon of the Duffing oscillator, but it is difficult for the HBM to get solutions for the higher-order harmonics. In the present study, when more harmonic components are taken into account in the application of the HBM, the software fails to find solutions. Therefore, although it has been claimed that the HBM is able to handle strongly nonlinear systems whose higher-order harmonics can make significant contributions to the system responses, in practice, due to computational limits the HBM is still not that powerful for analyzing complex nonlinear systems. That is why the HBM method cannot provide accurate predictions on the nonlinear damping oscillator (44). Moreover, it usually takes quite a long time for the HBM method to find the solutions, and it can be half an hour or even longer. The time used for the HBM method depends on the complexity of the nonlinear system under study and the number of the harmonics considered, for example, the HB3 can be accomplished in few minutes for Case 1 and, however, it took more than one hour for Case 2.

On the other hand, there is no computational limit for the NOFRF approach as the NOFRFs don't involve any equation solution procedure. However, the NOFRFs cannot capture the *jump phenomenon* of the Duffing

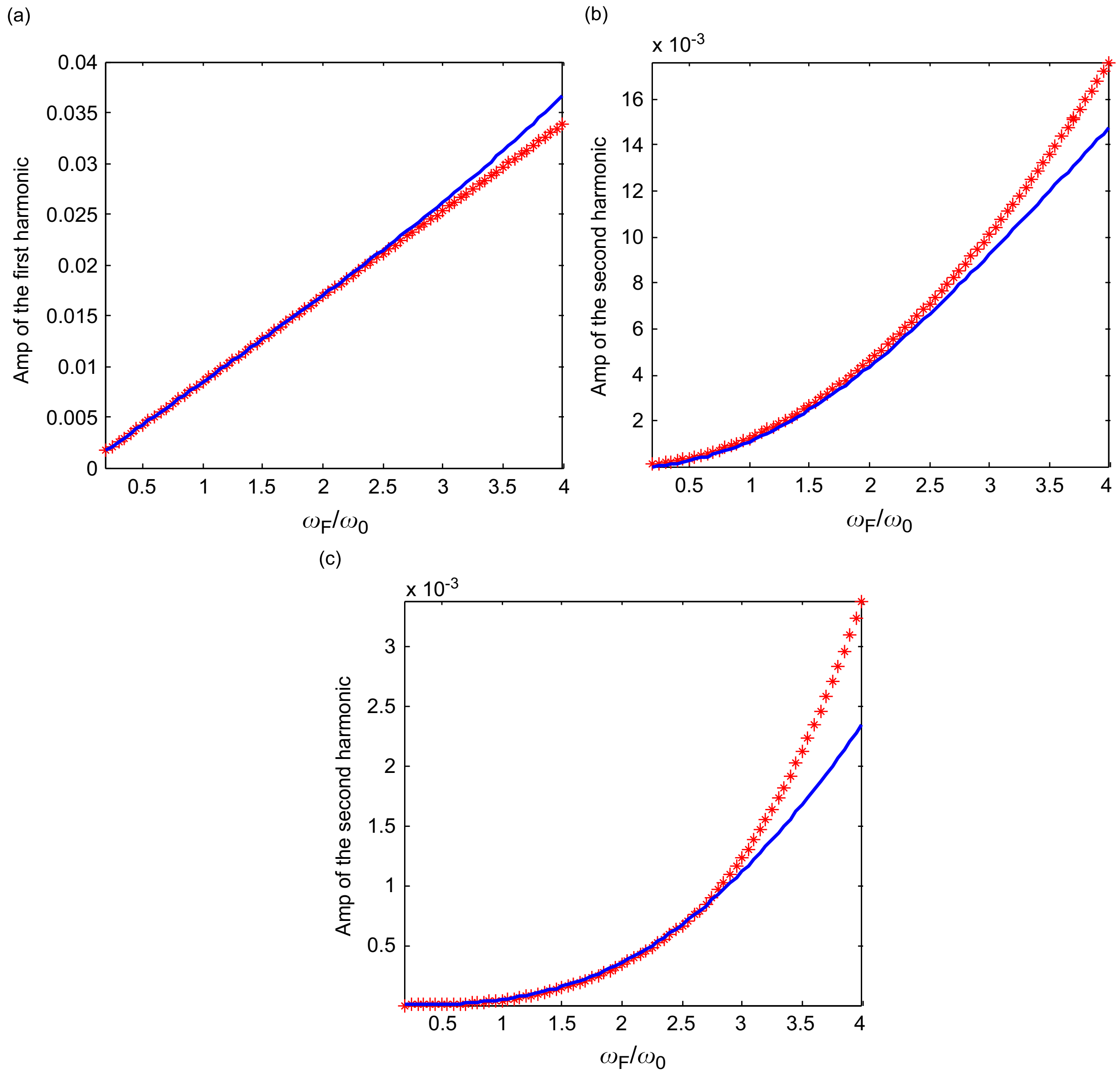

Fig. 8. Comparison between the HB3 and the *Runge–Kutta* method (star-HB3): (a) the first-order harmonic; (b) the second-order harmonic; and (c) the third-order harmonic.

oscillator. Although the occurrence of *jump phenomenon* usually indicates a strongly nonlinear behaviour, the inability of the NOFRF to capture the *jump phenomenon* does not mean that NOFRFs cannot handle other cases of strongly nonlinear systems. The case study for the nonlinear damping oscillator (44) shows that the NOFRF can accurately predict the response of this oscillator. As noted in Section 3, the Volterra series can describe the class of nonlinear systems which are stable at zero equilibrium. For some nonlinear systems like the Duffing oscillator, it is well-known that, for the input amplitude is over a certain value, and then a tiny change of the amplitude can introduce a large change in the behaviour of the nonlinear system. The NOFRF cannot handle this phenomenon because Volterra series theory, on which the NOFRF are based, cannot represent nonlinear systems in such situations. That is why the NOFRF cannot predict the motion of the Duffing oscillator (40) for the frequency range between $0.7\omega_0$ and $1.5\omega_0$. However, for some damping

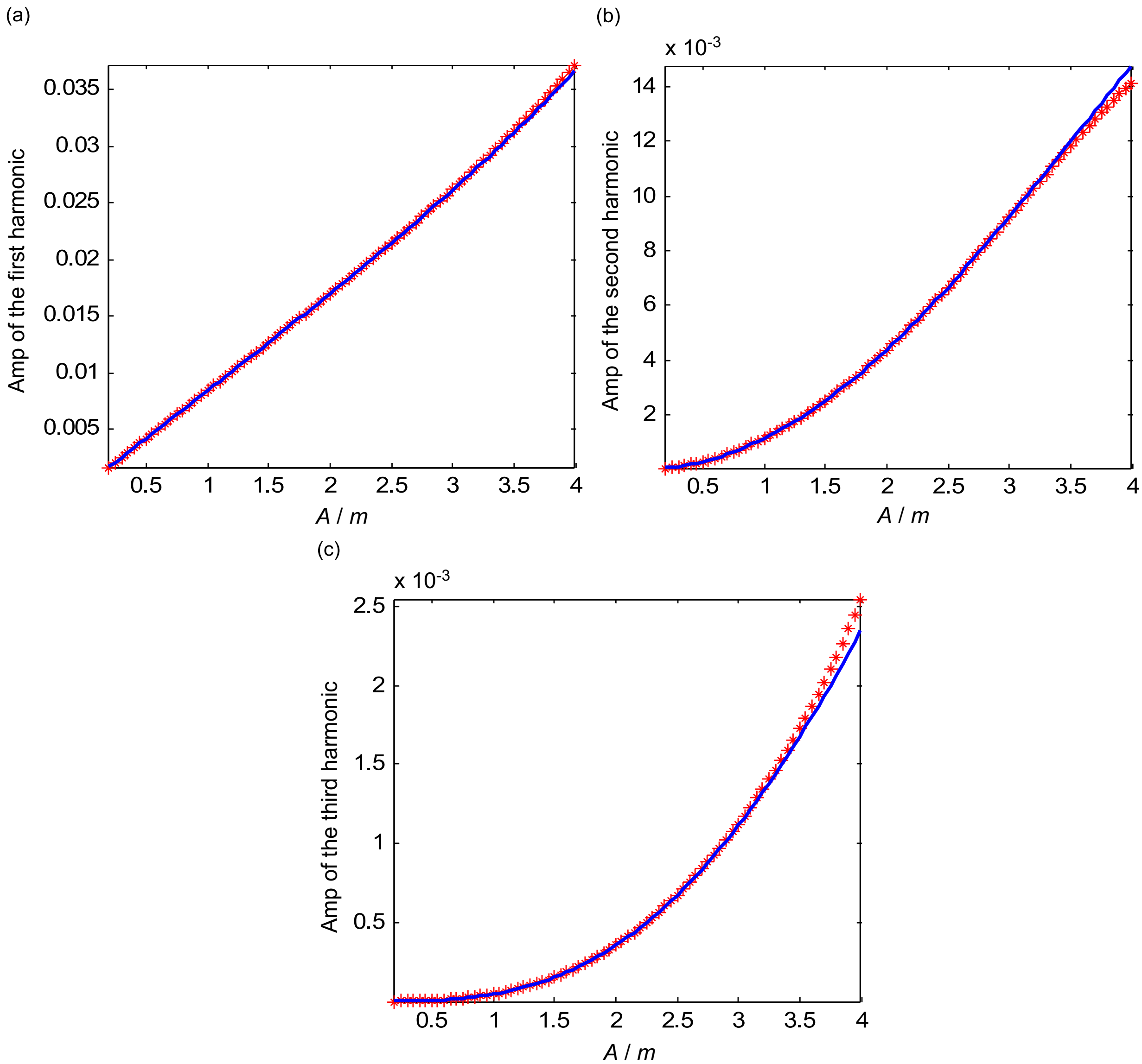

Fig. 9. Comparison between the NOFRF and the *Runge–Kutta* method (star-NOFRF): (a) the first-order harmonic; (b) the second-order harmonic; and (c) the third-order harmonic.

nonlinear systems like the oscillator (44), although the nonlinear damping can make the system behave strongly nonlinearly, it also makes the system behave more stable at zero equilibrium and thus helps the system to sustain stronger inputs. Fig. 10(a) shows the response and its spectrum obtained from a linear oscillator where m, k, c are the same as those used in Case 2. The frequency of the input is ω_0 (the natural frequency of the linear oscillator) and the amplitude of the input is $A/m = 3$. Fig. 10(b) shows the response and the spectrum of the oscillator (44) subjected to the same force. Obviously, the nonlinear damping force has caused the oscillator to behave strongly nonlinearly because the significant super-harmonics have appeared but, compared to the response of the linear oscillator, the vibration amplitude of the nonlinear oscillator has greatly decreased. This is essentially the principle used in nonlinear damping shock absorbers.

It is worth noting that the convergence regarding the Volterra series representation for nonlinear systems is a quite difficult and challenging problem. Great efforts [39,42–47] have been made to address this problem. But there are still no general criteria or methods available which can determine the convergence of the Volterra

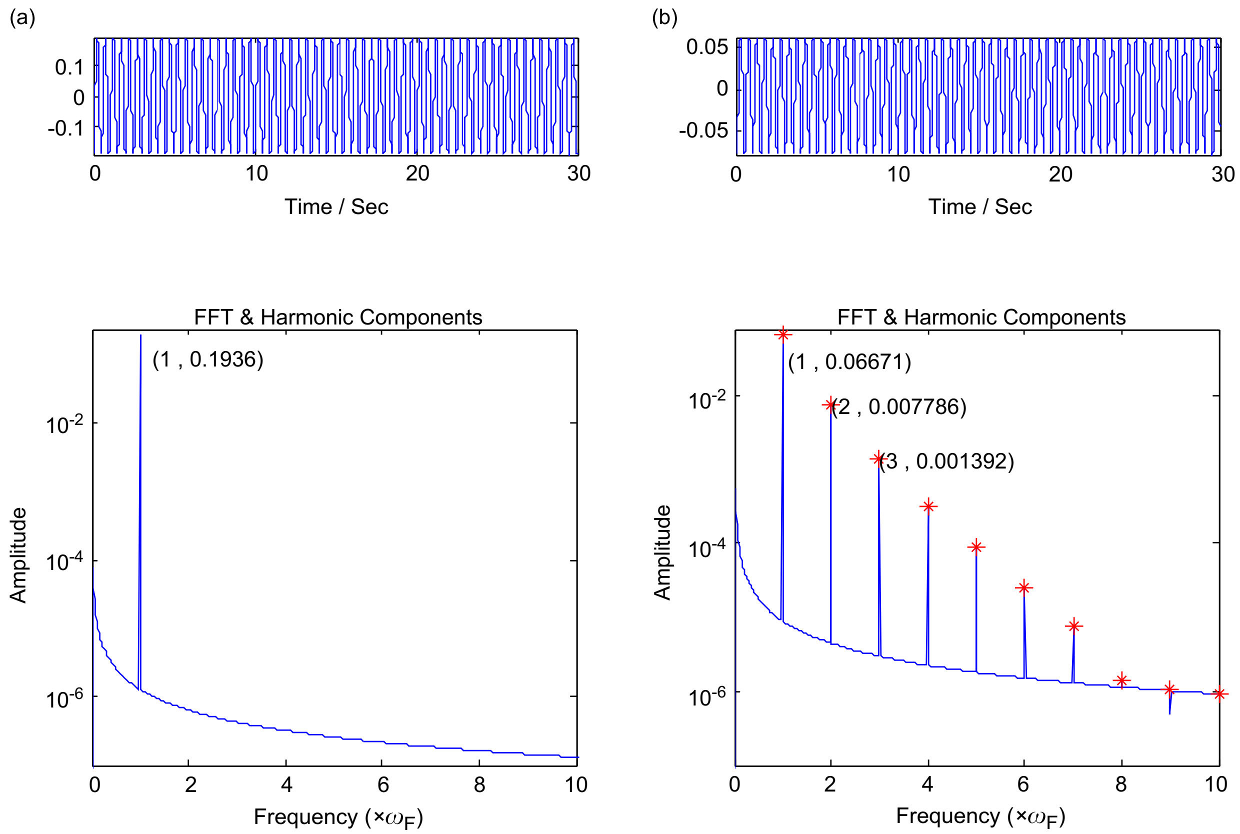

Fig. 10. Comparison between the output responses of a linear oscillator and a nonlinear oscillator: (a) response of linear oscillator and (b) response of nonlinear oscillator.

series representation for nonlinear systems. The available criteria are often very conservative and can only provide a rough estimation for the real convergence region. The only exception may be for the case of the Duffing oscillator subjected to a harmonic excitation. Three different criteria [39,44,45] are available and they can all accurately predict an upper limit on the amplitude of the harmonic excitation under which the Volterra series representation for the oscillator's response will converge.

6. Conclusions and remarks

The HBM is a well-established method for the analysis of nonlinear systems, the time domain response of which can be expressed as a Fourier series. The NOFRFs are a new concept proposed by the authors, which has been derived from the Volterra series and can be considered to be an extension of the classic FRF to the nonlinear case. When a nonlinear system is subjected to harmonic inputs, the system response can be directly expressed as a Fourier series using the NOFRFs. In this paper, using the well-known Duffing oscillator as a case study, the relationship between the HBM and the NOFRFs has been investigated. The results revealed that the harmonic components calculated using the NOFRFs are one of the solutions obtained using the HBM. The concept of the NOFRF has a solid theoretical basis—the Volterra series. The relationship which has been investigated in this study between the two methods should help researchers and engineers to understand the HBM and the NOFRF methods. The HBM is based on the assumption that the responses of the nonlinear systems consist of only harmonic components, but the method cannot explain why super-harmonics will appear when the nonlinear system is subject to a sinusoidal input. Even though the HBM can reveal sub-resonance phenomenon, for example the maximum at $\omega_F = 1/3\omega_0$ in Fig. 1(b), it cannot account for this nonlinear phenomenon. However, the NOFRFs can give a theoretical explanation for both the appearances of super-harmonics and sub-resonance; more details can be found in reference [35]. In addition, comparative studies using numerical methods have shown the HBM can capture the well-known *jump phenomenon*, but it will suffer from the computational limits. If more frequency components are taken into account in the HBM, it is highly possible for the software to fail to find solution. Therefore, for some strongly nonlinear systems, the HBM cannot provide accurate predictions of the harmonic components in the responses. The NOFRFs cannot capture *jump phenomenon* in the Duffing oscillator because the Volterra series theory doesn't work at the region around the jump point. But the NOFRFs does not suffer from the computational limits and can always be implemented in a few seconds. For some nonlinear systems, like the nonlinear damping oscillator (44), the NOFRFs can give much better predictions of the harmonic components compared to the HBM even if such systems are strongly nonlinear.

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